1. Exercise 1.2.1 on Page 8:

Find the union $C_1 \cup C_2$ and the intersection $C_1 \cap C_2$ of the two sets C_1 and C_2, where

(a) $C_1 = \{0, 1, 2\}$, $C_2 = \{2, 3, 4\}$

Answer(a):

$C_1 \cup C_2 = \{0, 1, 2, 3, 4\}$, $C_1 \cap C_2 = \{2\}$

(b) $C_1 = \{x : 0 < x < 2\}$, $C_2 = \{x : 1 \leq x < 3\}$

Answer(b):

$C_1 \cup C_2 = \{x : 0 < x < 3\}$, $C_1 \cap C_2 = \{x : 1 \leq < 2\}$

(c) $C_1 = \{(x, y) : 0 < x < 2, 1 < y < 2\}$, $C_2 = \{(x, y) : 1 < x < 3, 1 < y < 3\}$

Answer(c):

$C_1 \cup C_2 = \{(x, y) : 0 < x \leq 1, 1 < y < 2\} \cup \{(x, y) : 1 < x < 3, 1 < y < 3\}$

$C_1 \cap C_2 = \{(x, y) : 1 < x < 2, 1 < y < 2\}$

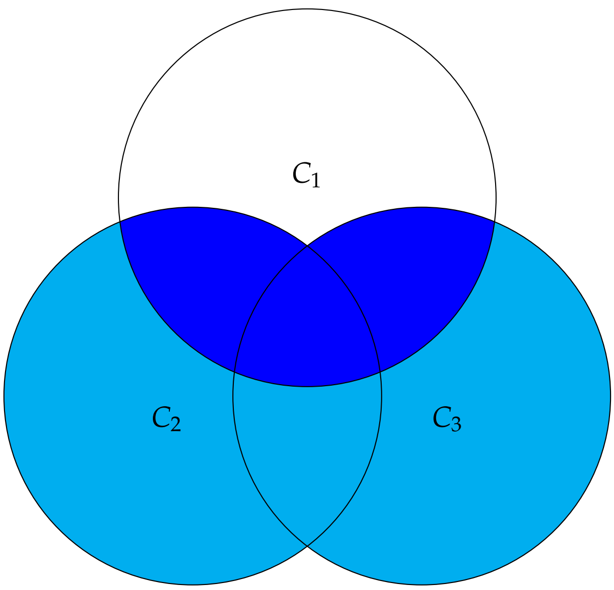
2. Exercise 1.2.1 on Page 8:

By the use of Venn diagrams, in which the space $\mathcal{C}$ is the set of points enclosed by a rectangle containing the circles C_1, C_2 and C_3, compare the following sets. These laws are called the distributive laws.

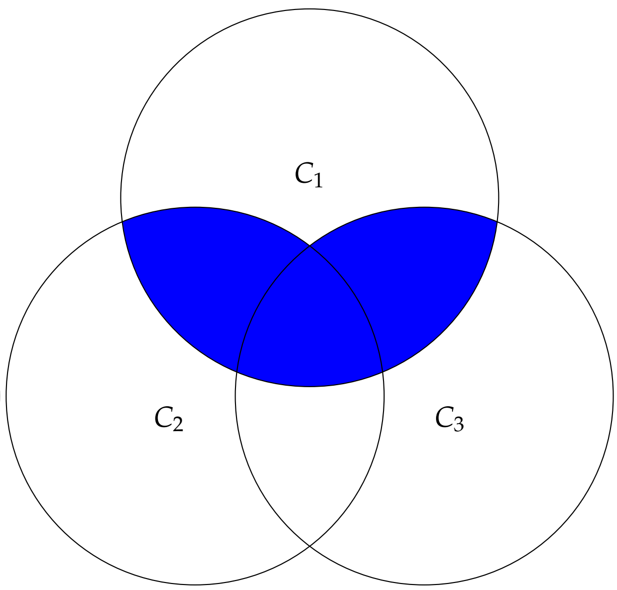
(a)$C_1 \cap (C_2 \cup C_3)$ and $(C_1 \cap C_2) \cup (C_2 \cap C_3)$

Answer(a):

$C_1 \cap (C_2 \cup C_3)$

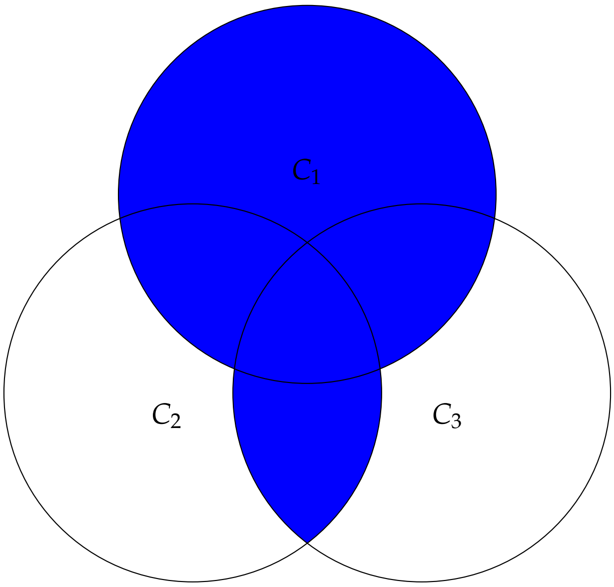


$(C_1 \cap C_2) \cup (C_2 \cap C_3)$

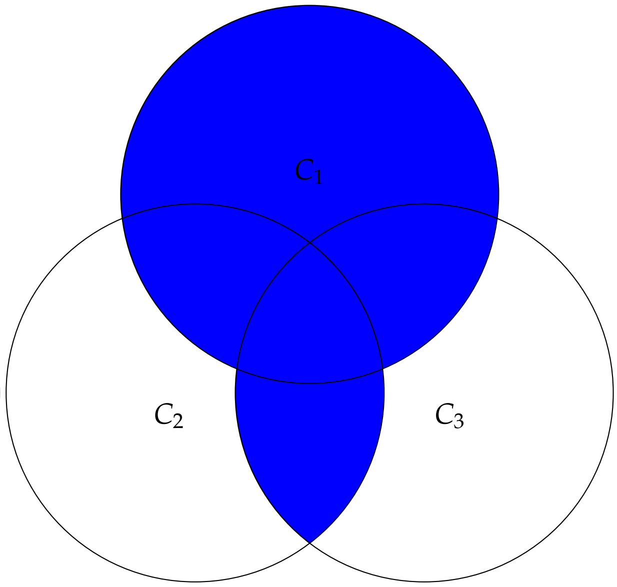


(b)$C_1 \cup (C_2 \cap C_3)$ and $(C_1 \cup C_2) \cap (C_1 \cup C_3)$

$C_1 \cup (C_2 \cap C_3)$



$(C_1 \cup C_2) \cap (C_1 \cup C_3)$



Formal Proof of Distributive Law:

Proof : $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Let $x \in A \cup (B \cup C)$. If $x \in A \cup (B \cup C)$ then x is either in A or in (B and C).

$x \in A$ or $x \in (B \text{ and } C)$

$x \in A$ or $\{x \in B \text{ and } x \in C\}$

$\{x \in A \text{ or } x \in B\}$ and $\{x \in A \text{ or } x \in C\}$

$x \in (A \text{ or } B)$ and $x \in (A \text{ or } C)$

$x \in (A \cup B) \cup x \in (A \cup C)$

$x \in (A \cup B) \cup (A \cup C)$

$x \in A \cup (B \cup C) => x \in (A \cup B) \cup (A \cup C)$

Thus $A \cup (B \cap C) \subset (A \cup B) \cap (A \cup C)$

Let $x \in (A \cup B) \cap (A \cup C)$. If $x \in (A \cup B) \cap (A \cup C)$ then x is in (A or B) and x is in (A or C).

$x \in$ (A or B) and $x \in$ (A or C)

$\{x \in A \text{ or } x \in B\}$ and $\{x \in A \text{ or } x \in C\}$

$x \in A$ or $\{x \in B \text{ and } x \in C\}$

$x \in A$ or $\{x \in (B \text{ and } C)\}$

$x \in A \cup \{x \in (B \cap C)\}$

$x \in A \cup (B \cap C)$

$x \in (A \cup B) \cap (A \cup C) => x \in A \cup (B \cap C)$

Therefore, $(A \cup B) \cap (A \cup C) \subset A \cup (B \cap C)$

In conclusion, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

3. Exercise 1.3.5 on Page 18:

Let the sample space be $\mathcal{C} = \{c : 0 < c < \infty\}$. Let $C \subset \mathcal{C}$ be defined by $C = \{c : 4 < c < \infty\}$ and take $P(C) = \int_C e^{-x}dx$. Show that $P(\mathcal{C}) = 1$. Evaluate $P(C)$, $P(C^c)$, and $P(C \cup C^c)$.
Answer:

$$P(\mathcal{C}) = \int_{\mathcal{C}} e^{-x}dx = \int_0^\infty e^{-x}dx = -e^{-x}|\{x=\infty\} - (-e^{-x}|\{x=0\}) = 0+1 = 1$$
$$P(C) = \int_C e^{-x}dx = \int_4^\infty e^{-x}dx = -e^{-x}|\{x=\infty\} - (-e^{-x}|\{x=4\}) = 0+e^{-4} = e^{-4}$$
$$P(C^c) = \int_{C^c} e^{-x}dx = \int_0^4 e^{-x}dx = -e^{-x}|\{x=4\} - (-e^{-x}|\{x=0\}) = -e^{-4}+1 = 1-e^{-4}$$
$$P(C \cup C^c) = P(\mathcal{C}) = 1$$

4. Exercise 1.3.14 on Page 19:

There are five red chips and three blue chips in a bowl. The red chips are numbered 1,2,3,4,5, respectively, and the blue chips are numbered 1,2,3, respectively. If two chips are to be drawn at random and without replacement, find the probability that these chips have either the same number or the same colour.
Answer:

$$P(\text{Same Number}) = \frac{\binom{3}{1}}{\binom{8}{2}} = \frac{3}{28}$$
$$P(\text{Same Colour}) = \frac{\binom{5}{2}+\binom{3}{2}}{\binom{8}{2}} = \frac{10+3}{28} = \frac{13}{28}$$

$$\begin{aligned}&P(\text{Same Number or Same Colour})\\ &= P(\text{Same Number}) + P(\text{Same Colour}) - P(\text{Same Number and Same Colour})\\ &= \frac{3}{28} + \frac{13}{28} - 0 = \frac{4}{7}\end{aligned}$$

5. Exercise 1.4.4 on Page 28:

From a well-shuffled deck of ordinary playing cards, four cars are turned over one at a time without replacement. What is the probability that the spades and red cards alternate?
Answer:
Method 1:
Two ways of rows:

1. Spade, Red, Spade, Red

$$\frac{13}{52} \times \frac{26}{51} \times \frac{12}{50} \times \frac{25}{49} = 0.01560624$$

2. Red, Spade, Red, Spade

Then,

$$\frac{26}{52} \times \frac{13}{51} \times \frac{25}{50} \times \frac{12}{49} = 0.01560624$$

Thus, $0.01560624 \times 2 = 0.03121248$.

Method 2:

$$Number(\text{Two Spades and Two Reds alternate}) = \binom{13}{2}\binom{26}{2} \times 4 \times 2$$

$$Number(\text{Choose Four Card with Order}) : P_4^{52}$$

$$P(\text{Two Spades and Two Reds alternate}) = \frac{\binom{13}{2}\binom{26}{2} \times 4 \times 2}{P_4^{52}} = 0.03121248$$

6. Exercise 1.4.27 on Page 31:

Each bag in a large box contains 25 tulip bulbs. It is known that 60% of the bags contain bulbs for 5 red and 20 yellow tulips, while the remaining 40% of the bags contain bulbs for 15 red and 10 yellow tulips. A bag is selected at random and a bulb taken at random from this bag is planted.

(a) What is the probability that it will be a yellow tulip?

Answer (a):

Define event A as yellow tulip is selected.

Define event B as 60% bags selected.

$$P(A) = P(A|B) \times P(B) + P(A|B^c) \times P(B^c) = 0.6 \times \frac{20}{5+20} + 0.4 \times \frac{10}{15+10} = \frac{16}{25}$$

(Using law of total probability)

(b) Given that it is yellow, what is the conditional probability it comes from a bag that contained 5 red and 20 yellow bulbs?

Answer(b):

$$P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{0.6 \times \frac{20}{5+20}}{\frac{16}{25}} = \frac{3}{4}$$

(Using Bayes Theorem or Conditional Probability)

7. Exercise 1.5.1 on Page 38:

Let a card be selected from an ordinary deck of playing cards. The outcome c is one of these 52 cards. Let $X(c) = 4$ if c is an ace, let $X(c) = 3$ if c is a king, let $X(c) = 2$ if c is a queen, let $X(c) = 1$ if c is a jack and let $X(c) = 0$ otherwise. Suppose that P assigns a probability of $\frac{1}{52}$ to each outcome c. Describe the induced probability $P_x(D)$ on the space $\mathcal{D} = \{0, 1, 2, 3, 4\}$ of random variable X.

Answer:

$$P_x(D) = \begin{cases} \frac{40}{52} = \frac{9}{13} & \text{X=0} \\ \frac{1}{13} & \text{X=1} \\ \frac{1}{13} & \text{X=2} \\ \frac{1}{13} & \text{X=3} \\ \frac{1}{13} & \text{X=4} \end{cases}$$

8. Exercise 1.5.8 on Page 39:

Given the c.d.f

$$F(x) = \begin{cases} 0 & x < -1 \\ \frac{x+2}{4} & -1 \leq x < 1 \\ 1 & 1 \leq x \end{cases}$$

sketch the graph of $F(x)$ and then compute:

(a) $P(-\frac{1}{2} < X \leq \frac{1}{2})$

Answer(a):

$$P(-\frac{1}{2} < X \leq \frac{1}{2}) = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x)dx = F(\frac{1}{2}) - F(-\frac{1}{2}) = \frac{1}{4}$$

(b)$P(X = 0)$

Answer (b):

$$P(X = 0) = 0$$

(c)$P(X = 1)$

Answer (c):

$$P(X = 1) = \frac{1}{4}$$

(d)$P(2 < X \leq 3)$

$$P(2 < X \leq 3) = F(3) - F(2) = 1 - 1 = 0$$

9. Exercise 1.6.7 on Page 43:

Let X have a p.m.f $p(x) = \frac{1}{3}$, $x = 1, 2, 3$, zero elsewhere. Find the p.m.f. of $Y = 2X + 1$.
Answer:

$$X = \frac{1}{2}Y - \frac{1}{2}$$

For the mappings,

$$\mathcal{D}_X = \{x : x = 1, 2, 3\} \qquad \mathcal{D}_Y = \{y : y = 3, 5, 7\}$$

Then,

$$P_Y(y) = P_X(\frac{1}{2}y - \frac{1}{2}) = \begin{cases} \frac{1}{3} & y = 3 \\ \frac{1}{3} & y = 5 \\ \frac{1}{3} & y = 7 \end{cases}$$

10. Exercise 1.7.6 on Page 50:

For each of the following p.d.f. of X, find $P(|X| < 1)$ and $P(X^2 < 9)$.
(a) $f(x) = x^2/18$, $-3 < x < 3$, zero else where.
Answer (a):

$$\begin{aligned} P(|X| < 1) = P(-1 < X < 1) = \int_{-1}^{1} f(x)dx = \int_{-1}^{1} (x^2/18)dx \\ = x^3/54|(x = 1) - x^3/54|(x = -1) \\ = \frac{1}{27} \\ P(X^2 < 9) = P(-3 < X < 3) = 1 \end{aligned}$$

(b) $f(x) = (x + 2)/18$, $-2 < x < 4$, zero elsewhere.
Answer (b):

$$\begin{aligned} P(|X| < 1) = P(-1 < X < 1) = \int_{-1}^{1} f(x)dx = \int_{-1}^{1} (x + 2/18)dx \\ = (\frac{1}{2}x^2 + 2x)/18|(x = 1) - (\frac{1}{2}x^2 + 2x)/18|(x = -1) \\ = \frac{2}{9} \end{aligned}$$

$$\begin{aligned}P(X^2 < 9) &= P(-3 < X < 3) = P(-2 < X < 3)\\ &= (\frac{1}{2}x^2 + 2x)/18|(x=3) - (\frac{1}{2}x^2 + 2x)/18|(x=-2)\\ &= \frac{25}{36}\end{aligned}$$

11. Exercise 1.8.8 on Page 57:

Let $f(x) = 2x, 0 < x < 1$, zero elsewhere, be the p.d.f. of X.

(a) Compute $\mathbb{E}(1/X)$

Answer (a):

$$\mathbb{E}(1/X) = \int_0^1 \frac{1}{x} f(x)dx = 2$$

(b) Find the c.d.f. and p.d.f. of $Y = 1/X$.

Answer (b):

Method 1:

$$P_Y(y) = P(Y \le y) = P(\frac{1}{X} \le y) = P(X > \frac{1}{y}) = \int_1^{1/y} 2xdx = 1 - \frac{1}{y^2}$$

Thus,

$$f_Y(y) = \frac{2}{y^3} \qquad y \in (0, \infty)$$

Method 2:

Jacobian of the transformation is

$$J = \frac{d\frac{1}{y}}{dy} = -\frac{1}{y^2}$$

$$\begin{aligned}f_Y(y) &= f_X(X = 1/y)(-dx/dy)\\ &= f_X(1/y)|J|\\ &= \frac{2}{y^3} \qquad y \in (0, \infty)\end{aligned}$$

(c) Compute $\mathbb{E}(Y)$ and compare this result with the answer obtained in part (a).

Answer (c):

$$\mathbb{E}(y) = \int_{y=0}^{\infty} y\frac{2}{y^3}dy = -\frac{2}{y}|(y=\infty) + \frac{2}{y}|(y=1) = 2$$

The results are consistent.