Lecture 2

2.6 Independence

	A	A'	
B	$\boxed{99}$	20	$\boxed{119}$
B'	10	401	411
	$\boxed{109}$	421	530

$$p(A \cap B) = p(A) \cdot p(B) \quad (1)$$

$$\Leftrightarrow p(A|B) = p(A) \quad (2)$$

$$\Leftrightarrow p(B|A) = p(B) \quad (3)$$

A = Patient indeed is sick

B = Doctor says he's sick.

$$p(A \cap B) = \frac{99}{530}$$

$$\neq$$

$$p(A) \cdot p(B) = \frac{109}{530} \cdot \frac{119}{530}$$

$\therefore$ A & B are dependent.

Example: In classroom,

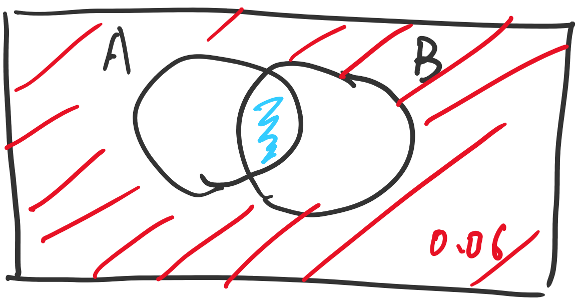
70% people, drive. $\Rightarrow A$

80% people. 3Y. $\Rightarrow B$

6% people neither A nor B.

$A' \cap B' = 0.06$

Are Driving and 3Y independent.



$p(A) = 0.7$

$p(B) = 0.8$

$\therefore p(A \cup B) = 1 - 0.06$

$= 0.94$

$\therefore$ Prop. of Union

$$p(A \cap B) = p(A) + p(B) - p(A \cup B)$$

$$= 0.7 + 0.8 - 0.94$$

$$= 0.56.$$

$$p(A) \cdot p(B) = 0.7 \cdot 0.8 = 0.56$$

$\therefore p(A \cap B) = p(A) \cdot p(B)$

A and B independent.

Mutually Exclusive:

$p(A \cap B) = 0$

Mutually Exclusive:

$$p(A \cap B) = 0$$

Independent

$$\underline{p(A \cap B)} = \underline{p(A)} \cdot \underline{p(B)}$$

2.8 Random Variables.

A function that assigns a real number to each outcome in the sample space.

$X \Leftarrow$	$\frac{1}{6}$	1
	$\frac{1}{6}$	2
	$\frac{1}{6}$	3 $\Longleftarrow$
	$\frac{1}{6}$	4
	$\frac{1}{6}$	5
	$\frac{1}{6}$	6

Notation. $X \Leftarrow$ Capital Letter R.V.

$x \Leftarrow$

$$p(X=1) = \frac{1}{6}$$

$$p(X=x) = \begin{cases} \frac{1}{6}, & x=1 \\ \frac{1}{6}, & x=2 \\ \frac{1}{6}, & x=3 \\ \frac{1}{6}, & x=4 \\ \frac{1}{6}, & x=5 \\ \frac{1}{6}, & x=6 \end{cases}$$

Discrete : A discrete R.V. is R.V with <u>finite</u> or <u>countable infinite</u> range.

Continuous :

$\underline{[1, 2]}$

1 . 2 4 . . . . . ∞

[1, 2]

$1, 2, 3, 4, \ldots \quad \infty$

$\frac{m}{n}$ rational number

→ RV with an interal [1, 2]

X denote the result of roll a dice. ⇐ D

Y denote height of a person. ⇐ C

Chapter 3

3.1

Example: 10 balls in an urn, 5 white, 5 Black.

(a) balls are drawn each time, without replacement, untill a black one is got.

Let X = # of draws needed.

white ... Black

$$\underline{P(X=4)} = \frac{5}{10} \cdot \frac{4}{9} \cdot \frac{3}{8} \cdot \frac{5}{7} = 0.0595$$

(b). what if we draw until two consecutive black balls are obtained? $\underline{P(X=5)}$?

B W W B B $\quad \frac{5}{10} \cdot \frac{5}{9} \cdot \frac{4}{8} \cdot \frac{4}{7} \cdot \frac{3}{6}$

W B W B B. $\quad \frac{5}{10} \cdot \frac{5}{9} \cdot \frac{4}{8} \cdot \frac{4}{7} \cdot \frac{3}{6}.$

W W W B B. $\quad \frac{5}{10} \cdot \frac{4}{9} \cdot \frac{3}{8} \cdot \frac{5}{7} \cdot \frac{4}{6}.$

$= 0.11905.$

$\therefore \underline{P(X=5) = 0.11905}$

$\therefore$ $P(X=3) = 0.1103$

The probality Distribution:

The probability distribution of R.V. X is a description of the probabilities associated with the possible values of X.

$$p(X=x) = f(x) = \begin{cases} \frac{1}{6}, & x=1 \\ \frac{1}{6}, & x=2 \\ \frac{1}{6}, & x=3 \\ \frac{1}{6}, & x=4 \\ \frac{1}{6}, & x=5 \\ \frac{1}{6} & x=6 \end{cases} \Leftarrow \text{P.M.F}$$

$p(X=7)=0$

Probability Mass Function. P.M.F.

(1) $P(X=x) = f(x)$

(2) $f(x_i) \geq 0$

(3) $\sum_{S} f(x_i) = 1$

Example: 200 students, 40 3J, 160 3Y.

I randomly choose 3 students.

X denotes the # of 3J in this 3.

Write down P.M.F for X.

$$P(X=x) = f(x) = \begin{cases} \dfrac{\binom{40}{0}\binom{160}{3}}{\binom{200}{3}} = 0.5101, & X=0 \\ \dfrac{\overset{1.\,3J}{\binom{40}{1}}\;\overset{2,\,3Y}{\binom{160}{2}}}{\binom{200}{3}} = 0.3874, & X=1 \\ \dfrac{\binom{40}{2}\binom{160}{1}}{\cdots} = 0.0950, & X=2 \end{cases}$$

$$\begin{cases} \dfrac{\binom{40}{2}\binom{160}{1}}{\binom{200}{3}} = 0.0950, & x=2 \\ \dfrac{\binom{40}{3}\binom{160}{0}}{\binom{200}{3}} = 0.0075, & x=3 \end{cases}$$

Cumulative Distribution Function: C.D.F.

$F(x)$ or $\underline{P(X \le x)}$

Definition: $F(x_i) = P(X \le x_i) = \sum_{x \le x_i} f(x)$

Properties:

(1) $F(X) = P(X \le x) = \sum_{x < x_i} f(x)$

(2) $0 \le F(x) \le 1$

(3) If $x \le y$, then $F(x) \le F(y)$.

Example: 200 students, 40 3J, 160 3Y,

$\therefore$ CDF:

$$P(X \le x) = \begin{cases} 0.5101 & x=0 \\ 0.5101 + 0.3874 = 0.8975, & x=1 \\ 0.5101 + 0.3874 + 0.095 = 0.9925 & x=2. \\ 0.5101 + \cdots + 0.0075 = 1 & x=3. \end{cases}$$

3.4 Mean and Variance of a Discrete R.V.

Mean or Expected Value

$$\mu = E(x) = \sum_x x f(x)$$

Variance

$$\sigma^2 = V(x) = E[(X-\mu)^2] = \sum (x-\mu)^2 f(x).$$

Variance

$$\sigma^2 = \underline{V(X)} = E\left[(X-\mu)^2\right] = \sum_{x} (x-\mu)^2 f(x).$$

Example: PMF

$$P(X=x) = \begin{cases} \frac{1}{6} & x=1 \\ \vdots & \vdots \\ \frac{1}{6} & x=6. \end{cases}$$

$\Leftarrow$ $f(1) = P(X=1) = \frac{1}{6}$

Mean: $\mu = E(X) = \underset{f(1)}{\frac{1}{6}} \cdot \underset{1}{1} + \underset{f(2)}{\frac{1}{6}} \cdot \underset{2}{2} + \frac{1}{6} \cdot 3 + \frac{1}{6} \cdot 4 + \cdots \frac{1}{6} \cdot 6.$

$\cdots\cdots$ $= 3.5.$

Variance

$$\sigma^2 = V(X) = \underset{f(1)}{\frac{1}{6}} \underset{(1-\mu)^2}{(1-3.5)^2} + \underset{f(2)}{\frac{1}{6}} \underset{(2-\mu)^2}{(2-3.5)^2} + \cdots \frac{1}{6}(6-3.5)^2$$

$= 2.91667.$

Standard Deviation:

$$\sigma = \sqrt{2.91667} = 1.7078.$$

Example:

$$P(X=x) = f(x) = \begin{cases} 0.5101, & x=0 \\ 0.3874, & x=1 \\ 0.0950, & x=2 \\ 0.0075, & x=3. \end{cases}$$

Mean: $\mu = E(X) = 0.5101 \cdot 0 + 0.3874 \cdot 1 + 0.0950 \cdot 2$

$+ 0.0075 \cdot 3$

$= 0.6 \leftarrow \quad (0.5999)$

$= 0.6 \leftarrow \quad (0.5999)$

Variance:

$$\sigma^2 = V(X) = 0.5101(0-0.6)^2 + 0.3874(1-0.6)^2 + \cdots 0.0075(3-0.6)^2$$
$$= 0.4750$$

$$\boxed{V(X) = E(X^2) - [E(X)]^2.}$$

$$E(X^2) = 0.5101 \cdot 0^2 + 0.3874 \cdot 1^2 + 0.095 \cdot 2^2 + 0.0075 \cdot 3^2 = 0.8349.$$

$$\therefore V(X) = E(X^2) - [E(X)]^2 = 0.4750.$$

Binomial Distribution

Bernouli Trial: a trial with two possible outcomes.

Flip a coin.

Roll a dice and see where it is 6.

$\Downarrow$ repeat n times.

Binomial Distribution:

A random experiment consist of n Bernouli trials:

(1) Trials are independent.

(2) Each trial results in only two possible outcomes, labeled as success or failure.

(3) The probability of success is 'p'. consistent.

The R.V X equals the # of trials

The R.V X equals the # of trials that successes. with $0<p<1$ and $n=1,2\cdots\infty$.

X P.M.F :

x out of n to be head.

$$P(X=x)=f(x)=\binom{n}{x}p^x(1-p)^{n-x}, \quad x=1,2\cdots n.$$

$$\overbrace{\boxed{H}\quad\boxed{T}\quad\boxed{H}\quad\cdots\cdots\quad\boxed{T}}^{n \text{ times}}$$

$p \quad (1-p) \quad p \qquad\qquad (1-p)$

$p \Leftarrow$ Head

$1-p \Leftarrow$ Tail.

# of head : x.

$$\binom{n}{x}\underbrace{p\cdots\cdots p}_{x}\ \underbrace{(1-p)\cdots(1-p)}_{n-x}$$

$$p^x \qquad (1-p)^{n-x}$$

Example.

Suppose, 10% people are left handed. Now, we have a sample of 5 students. Let X denotes the # of left handeded students in this sample of 5.

(a) Write down the P.M.F of X

$$P(X=x)=\begin{cases}\binom{5}{0}0.1^0\,0.9^5 & x=0\\ \binom{5}{1}0.1^1\,0.9^4 & x=1\\ \binom{5}{2}0.1^2\,0.9^3 & x=2\\ \binom{5}{3}0.1^3\,0.9^2 & x=3\\ \binom{5}{4}0.1^4\,0.9^1 & x=4\\ \binom{5}{5}0.1^5\,0.9^0 & x=5\end{cases}$$

$$\left| \begin{array}{llll} \binom{5}{4} & 0.1^4 & 0.9^1 & X=4 \\ \binom{5}{5} & 0.1^5 & 0.9^0 & X=5. \end{array} \right|$$

(b). What is the probability that we have at least 1 left-handed?

$$P(X \geq 1) = 1 - P(X=0).$$

(c) Mean and variance of X.

$$\therefore \mu = E(X) = 0 \cdot Pr(X=0) + 1 \cdot Pr(X=1) + 2 \cdot Pr(X=2) + \cdots 5 \cdot Pr(X=5)$$

$$= 0.5$$

$$= \underset{5}{n} \cdot \underset{0.1}{p}$$

$$\mu = E(X) = n \cdot p$$

$$E(X^2) = 0^2 \cdot Pr(X=0) + 1^2 \cdot Pr(X=1) + 2^2 \cdot Pr(X=2) + \cdots + 5^2 Pr(X=5)$$

$$= 0.7.$$

$$V(X) = E(X^2) - [E(X)]^2 = 0.45. = \underset{5}{n}\,\underset{0.1}{p}\,(1-\underset{0.9}{p)}$$

$$V(X) = np(1-p)$$

$$(a+b)^n = \binom{n}{0} a^0 b^n + \binom{n}{1} a^1 b^{n-1} + \cdots \binom{n}{n-1} a^{n-1} b^1 + \binom{n}{n} a^n b^0$$

$(a+b)^n$ ← left; $+ \binom{n}{n} a^n b^0$ ⇐ right.

$a=1, \quad b=1$

Left hand side:

Left hand side:

$(1+1)^n = 2^n$.

n coins

$$\overbrace{\square \quad \square \quad \square \quad \cdots\cdots \quad \square}^{n}$$

$$2 \cdot 2 \cdot 2 \cdots\cdots 2$$

$$= 2^n$$

Right hand side:

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots\cdots + \binom{n}{n-1} + \binom{n}{n}$$

no coin is H — one coin is H — 2 coins are H. — n coins are H.

Geometric Distribution:

In a series of Bernoulli trials.

(Independent, with p of success constant)

The R.V X equals <u>\# of trials</u> until the first success happens.

$$P(X=x) = f(x) = \underbrace{(1-p)^{x-1}\, p}, \qquad x = 1, 2, \cdots\cdots \infty$$

$$\underbrace{\underbrace{\boxed{T} \quad \boxed{T} \quad \boxed{T} \quad \cdots\cdots}_{(1-p)^{x-1}} \quad \boxed{H}}_{x}$$

$$(1-p)^{x-1} \cdot p$$

Example:

Roll a dice, until the first '6' appear.

x = # of Rolls we made.

$p = \frac{1}{6}$

$p = \frac{1}{6}$

$$P(X=x) = \begin{cases} \frac{1}{6} & , x=1 \\ \frac{5}{6}\cdot\frac{1}{6} & , x=2 \\ \frac{5}{6}\cdot\frac{5}{6}\cdot\frac{1}{6} & , x=3 \\ \vdots & \vdots \\ \left(\frac{5}{6}\right)^{k-1}\cdot\frac{1}{6} & ; x=k \end{cases}$$

(a) Probability that I have a "6" within 2 and 4 inclusive.

$$P(X=2,3\text{ or }4) = P(X=2)+P(X=3)+P(X=4)$$
$$= \frac{5}{6}\cdot\frac{1}{6} + \left(\frac{5}{6}\right)^2\frac{1}{6} + \left(\frac{5}{6}\right)^3\frac{1}{6}$$

(b) Mean, Variance

$\left(\frac{5}{6}\right)^{k-2}\frac{1}{6}(k-1)$

$$\mu = E(X) = \frac{1}{6}\cdot 1 + \frac{5}{6}\cdot\frac{1}{6}\cdot 2 + \left(\frac{5}{6}\right)^2\frac{1}{6}\cdot 3 + \cdots\cdots \left(\frac{5}{6}\right)^{k-1}\frac{1}{6}\cdot k$$
$+ \cdots\cdots$

① $A =$

② $\frac{5}{6}A = \quad 0 + \frac{5}{6}\cdot\frac{1}{6}\cdot 1 + \left(\frac{5}{6}\right)^2\cdot\frac{1}{6}\cdot 2 +$

$+ \square$

$\left(\frac{5}{6}\right)^k\frac{1}{6}(k+1)$

$\left(\frac{5}{6}\right)^{k-1}\frac{1}{6}\cdot(k-1)$

$+\left(\frac{5}{6}\right)^k\frac{1}{6}(k)$

LHS $= \frac{1}{6}A$

RHS $= \frac{1}{6}\cdot 1 + \frac{5}{6}\frac{1}{6}(2-1) + \left(\frac{5}{6}\right)^2\frac{1}{6}(3-2) + \cdots\cdots$

$\left(\frac{5}{6}\right)^{k-1}\frac{1}{6}\left[k-(k-1)\right]$

$+ \cdots\cdots\cdots$

$$= \frac{1}{6}\left[1 + \frac{5}{6} + \left(\frac{5}{6}\right)^2 + \cdots\cdots\right]$$

$$= \frac{1}{6}\left[1 + \frac{5}{6} + \left(\frac{5}{6}\right)^2 + \cdots\right] + \cdots$$

$$= \frac{1}{6} \cdot \frac{1}{1-\frac{5}{6}}$$

$$= 1. \qquad p = \frac{1}{6}.$$

$$\therefore \frac{1}{6} \cdot E(X) = 1$$

$$\therefore E(X) = 6 = \frac{1}{p} \qquad \mu = E(X) = \frac{1}{p}$$

$$E(X^2) = \frac{1}{6} \cdot 1 + \frac{5}{6} \cdot \frac{1}{6} \cdot 2^2 + \left(\frac{5}{6}\right)^2 \frac{1}{6}\, 3^2 + \cdots$$

$$= 66$$

$$\therefore \sigma^2 = V(X) = 30.$$

$$\sigma^2 = E(X) = \frac{1-p}{p^2}$$

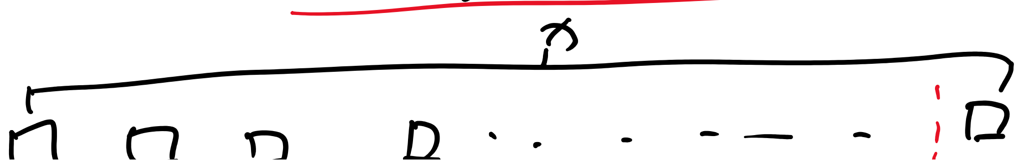
Negative Binomial Distribution:

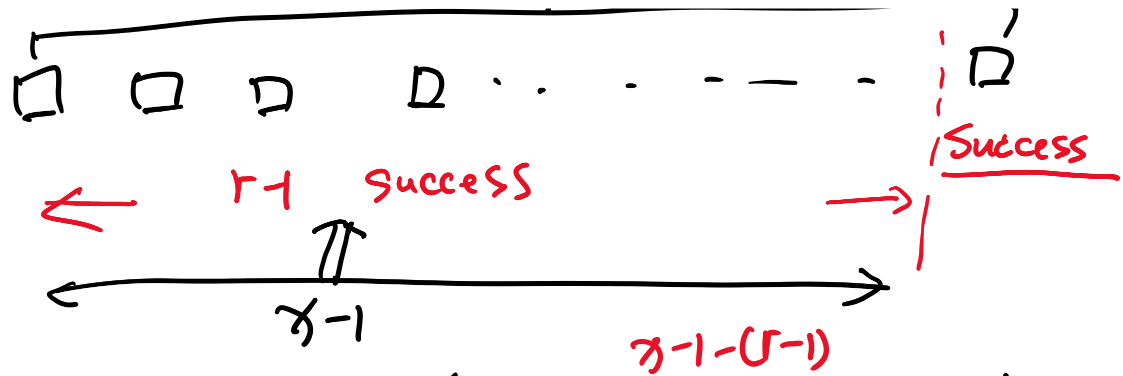
In a series of Bernoulli trials (Independent, with constant p).

X denotes # of trials until r success happens.

when $r=1$: $(1-p)^{x-1} p$

$$P(X=x) = \binom{x-1}{r-1}(1-p)^{x-r} p^r. \qquad x = r, r+1, \cdots \infty$$





$$\left[\binom{x-1}{r-1} p^{r-1} (1-p)^{x-r} \cdot \right] p$$

r-1 success of x-1 trials.

Example:

Roll a dice until 3 '6' occur.

X = # of trials until we get 3 '6'.

(a) p.m.f of X:

$$P(X=x)=\begin{cases} \frac{1}{6}\cdot\frac{1}{6}\cdot\frac{1}{6}, & x=3 \\ \binom{3}{2}\left(\frac{1}{6}\right)^3\cdot\frac{5}{6}, & x=4 \\ \binom{4}{2}\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^2, & x=5 \\ \vdots & \vdots \end{cases}$$

$$P(X=5)=\binom{4}{2}\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^2$$

(b) Find Mean and Variance.

$r=3$

(b) Find Mean and Variance.

$$\mu = E(X) = \frac{r}{p} = \frac{3}{\frac{1}{6}} = 18 \qquad \begin{cases} r = 3 \\ p = \frac{1}{6} \end{cases}$$

$$\sigma^2 = V(X) = \frac{r(1-p)}{p^2} = \frac{3(1-\frac{1}{6})}{\frac{1}{36}} = 90$$

Hypergeometric Distribution

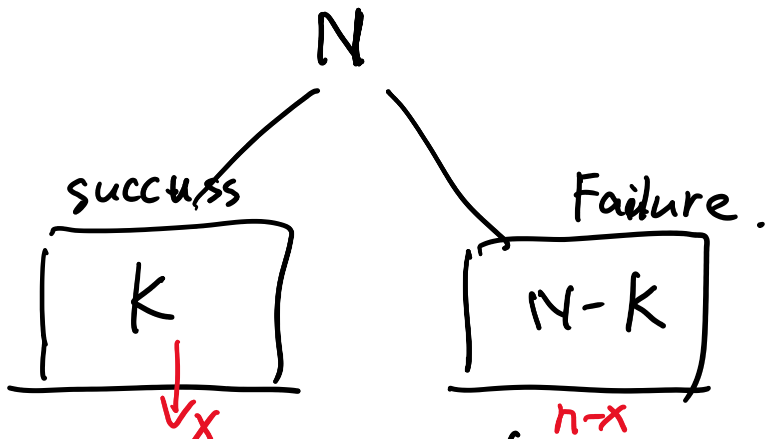
(without replacement)

A set of N objects, contains K objects classied as success. $N-K$ objects are failures.

A sample of size n objects randomly chosen. without replacement from the N objects.

where $K \le N$, $n \le N$.

X = # of successes.



choose n out of N

X = # of success from n choices.

$P(X = x) = \binom{K}{\ }\binom{N-K}{\ }$

$$P(X=x) = \frac{\binom{K}{x}\binom{N-K}{n-x}}{\binom{N}{n}} \Leftarrow$$

Example: 200 student 40 3J, 160 3Y.

X = # of 3J students.

$N=200$, $K=40$, $n=3$

$$P(X=x) = \begin{cases} \dfrac{\binom{40}{0}\binom{160}{3}}{\binom{200}{3}}, & x=0 \\ \dfrac{\binom{40}{1}\binom{160}{2}}{\binom{200}{3}}, & x=1 \\ \quad '' , & x=2 \\ \quad '' & x=3 \end{cases}$$

200 40 3J 160 3Y

if choose 3 with replacement.

$$\underbrace{n=3 \qquad p=\frac{40}{200}=0.2 \qquad 1-p=\frac{160}{200}=0.8.}_{\text{Binomial}}$$

$$P(X=x)=\binom{3}{x}\,\underline{0.2}^{\underline{x}}\;\underline{0.8}^{\underline{3-x}}, \quad x=0,1,2,3.$$

x out of 3, 3J students

Mean and Variance for Hypergeometric Distribution:

Mean: $\mu = E(X) = np = 0.6$, $\quad p = \dfrac{K \leftarrow \text{\# success}}{N \leftarrow \text{\# total}}$

Mean: $\mu = E(X) = np = 0.6$, $N \leftarrow \#\text{ total}$

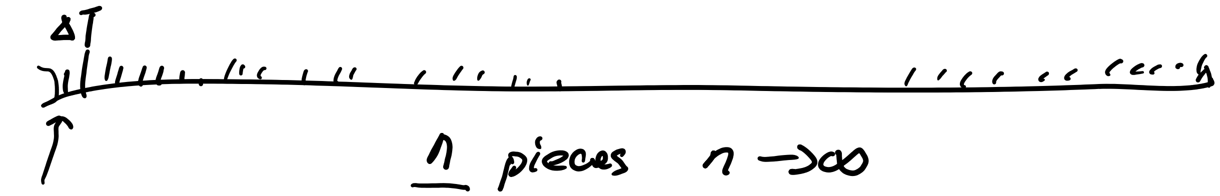
Variance:

$$\underbrace{\sigma^2 = V(X) = np(1-p)\cdot\frac{N-n}{N-1}} = 0.4750$$

Poisson Distribution:

Example: The average email I get per day is 10.

What is probability that I get 20 emails tomorrow? A Day.



ΔT $\vdash\!\dashv$ $p \leftarrow$ probability 2 get an email during ΔT

20 success.

$$\underbrace{p = \frac{10}{n}} \Leftarrow \Delta T$$

$$P(X=x) = \binom{n}{x} p^x (1-p)^{n-x}$$, $x = 20$ then we solve the problem.

$$= \boxed{\binom{n}{x}\left(\frac{10}{n}\right)^x}\left(1-\frac{10}{n}\right)^{n-x}, \quad n \to \infty$$

$$\left(1-\frac{10}{n}\right)^n \cdot \left(1-\frac{10}{n}\right)^{-x}$$

$$\Rightarrow \lim_{n\to\infty}\left(1+\frac{10}{n}\right)^n = e^{10}$$

$$\frac{n!}{(n-x)!} \quad \cdots [n-(x+1)]$$

$n \to \infty$

$$\binom{n}{x} = \frac{n!}{\ } = \frac{[n(n-1)\ldots(n-x-1)]}{x!}$$

$$\binom{n}{x} = \frac{\cancel{n!}}{\cancel{(n-x)!}\,x!} = \frac{n(n-1)\cdots(n-x-1)}{x!}$$

$$\left(\frac{10}{n}\right)^x = 10^x \cdot \left(\frac{1}{n}\right)^x = 10^x \cdot \frac{1}{\underbrace{n \cdot n \cdot n \cdots n}_{x}}$$

$$\left(1-\frac{10}{n}\right)^n = e^{-10}$$

$$\lim_{n\to\infty}\left(1-\frac{10}{n}\right)^{-x} = 1$$

$$\lim_{n\to\infty} = \frac{n(n-1)\cdots(n-x-1)}{n \cdot n \cdots n} \cdot \frac{10^x e^{-10}}{x!}$$

$$= \frac{10^x e^{-10}}{x!}$$

$x = 20.$

$$P(X=20) = \frac{10^{20} e^{-10}}{20!}$$

which is the probability that I will get 20 emails, given the averge emails/day is 10.

Poisson Distribution

P.M.F.

$$P(X=x) = \frac{e^{-\lambda}\lambda^x}{x!}$$ where λ is the averge of X.

$x = 0, 1, 2 \ldots\ldots$

$\mu = E(X) = \boxed{\lambda}$

$\sigma^2 = V(X) = \lambda$

$\sigma^2 = V(X) = \lambda.$

Example:

The # of people in a hospital is 3 in a given period.

Assume # of people in it follows Poisson Distribution. X denotes # of people in that period.

P.M.F of X. $3^0 = 1 \quad 0! = 1$

$$P(X=x) = \begin{cases} \frac{e^{-3}3^0}{0!} = e^{-3}, & x=0 \\ \frac{e^{-3}3^1}{1!}, & x=1 \\ \frac{e^{-3}3^2}{2!}, & x=2 \\ \frac{e^{-3}3^3}{3!}, & x=3 \\ \vdots & \vdots \end{cases}$$

(a) At least 1 patient is there.

$$P(X \geq 1) = 1 - P(X=0)$$
$$= 1 - e^{-3}$$

(b) Mean and Variance of X.

$$E(X) = \mu = \frac{e^{-3} \cdot 3^0}{0!} \cdot 0 + \frac{e^{-3} \cdot 3^1}{1!} \cdot 1 + \frac{e^{-3} \cdot 3^2}{2!} + \cdots$$
$$+ \cdots \underbrace{\frac{e^{-3} 3^x}{x!} \cdot x}_{x\text{-th}} + \cdots$$

$$= e^{-3}\underbrace{\left[0 + \frac{3}{1!} + \frac{3^2}{2!} + \frac{3^3}{3!} + \cdots \overset{x\text{-th}}{\frac{3^x}{x!}} + \cdots\right]}_{\text{taylor expansions.}}$$

$$= e^{-3} e^{3} \cdot 3$$

$$= 3 = \lambda.$$

$$\boxed{\begin{aligned} E(X) &= \mu = \lambda \\ Var(X) &= \sigma^2 = \lambda. \end{aligned}}$$