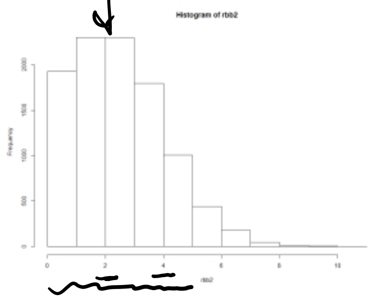
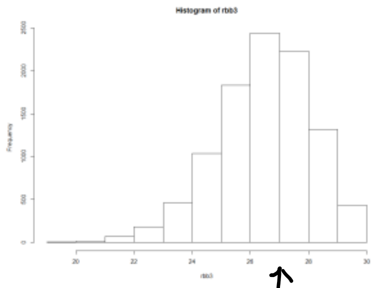
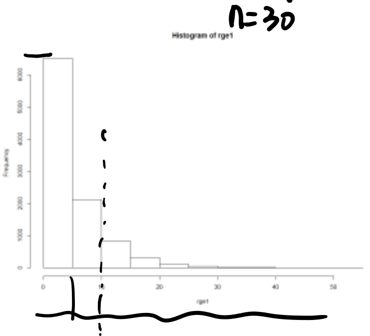
$p=0.5$ $\quad$ $p=0.1$ $\quad$ $p=0.9$.





Binomial: (n, p)

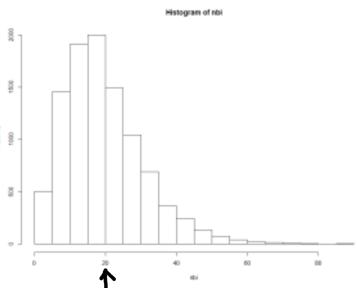
$n=30$ $\quad$ $n=30$ $\quad$ $n=30$.



$p=\frac{1}{6}$

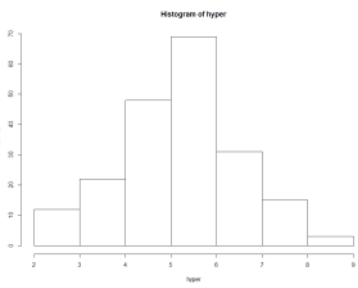
10,000

6000 samples out 10,000
stop before 5 draws.

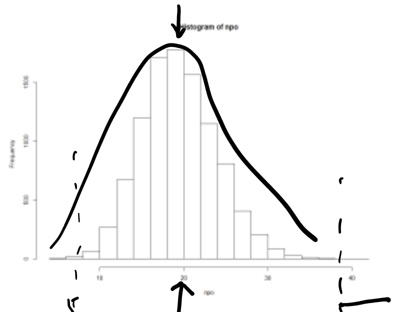


$r=4$
$p=\frac{1}{6}$.
10,000

$E(x) = \frac{r}{p} = 24$.

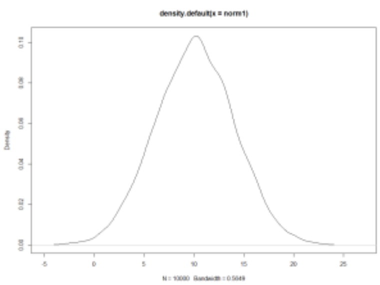


0.5

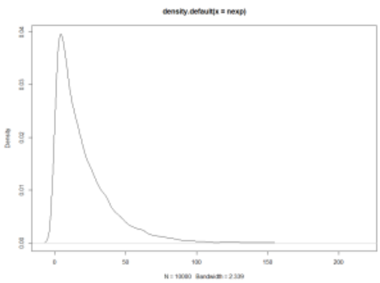


poisson $(\lambda=20)$.

Normal distribution.



Normal distribution.



Expontial

$\lambda = 20$ # custom / hour

$\frac{1}{\lambda} = \frac{1}{20}$ hour / custom

time

Chapter: Continuous. R.V.

Recall Discrete $\Rightarrow$ P.M.F (Mass)

Continuous $\Rightarrow$ P.D.F (Density.)

Probability Density Function:

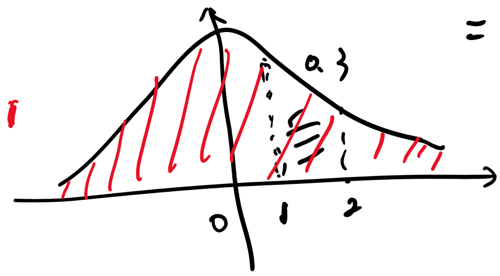
Continuous R.V X. p.d.f.

(1) $f(x) \geq 0$

(2) $\int_{-\infty}^{\infty} f(x)\,dx = 1 \iff \sum f(x) = 1$

(3) $P(a \leq X \leq b) = \int_a^b f(x)\,dx$

= area under $f(x)$ from a to b.



$\boxed{P(1) = 0}$

$P(1 \leq X \leq 2) = 0.3$.

Note: A histogram is an approximation to a probability density function.

The p.d.f $f(x)$ is used to calculate

The p.d.f $f(x)$ is used to calculate the area that represents the probability.

$p(X \le 1) = p(X < 1)$ if X is continuous.

Example:

Let X denotes the diameter of hole.

The target is 12.5 millimeters.

p.d.f is

$$f(x) = 20\, e^{-20(x-12.5)}, \quad x \geq \boxed{12.5}$$

Then Find $p(X > 12.6)$

$$p(x > 12.6) = \int_{12.6}^{\infty} f(x)\, dx$$

$$= \int_{12.6}^{\infty} 20\, e^{-20(x-12.5)}\, dx$$

$$= -e^{-20(x-12.5)} \Big|_{12.6}^{\infty}$$

$$= 0.135$$

$$p(12.5 < X < 12.6) = 1 - 0.135 = 0.865.$$

$$= \int_{12.5}^{12.6} f(x)\, dx = 0.865.$$

Example:

$$f(x) = \frac{c e^{-2(x-4)}}{1 + e^{-2(x-4)}}, \quad x \geq 4.$$

(a) value of c ?

$$\int_{4}^{\infty} f(x)\, dx = 1.$$

$$\int^{\infty} \frac{c e^{-2(x-4)}}{\quad} dx$$

$$\int_4^\infty \frac{c e^{-2(x-4)}}{1+e^{-2(x-4)}}\, dx.$$

$u = e^{-2(x-4)}$, then $du = -2e^{-2(x-4)} dx.$

$$= \int \frac{c\left(-\frac{1}{2}\right)}{1+u} du$$

$$= -\frac{1}{2}\cdot c \ln\left(1+e^{-2(x-4)}\right)\Big|_4^\infty$$

$$= 1$$

$$\therefore\ c = \frac{2}{\ln 2}$$

(b). $f(x)$ p.d.f for diameter for a hole.

drill 30 holes, what is mean # of holes that have a diameter greater than 5.

$$P(X \geq 5) = \int_5^\infty \frac{2}{\ln 2} \cdot \frac{e^{-2(x-4)}}{1+e^{-2(x-4)}} dx$$

for a single hole

$$= \frac{2}{\ln 2}\left[-\frac{1}{2}\ln\left[1+e^{-2(x-4)}\right]\right]\Big|_5^\infty$$

$$= \frac{1}{\ln 2}\left[\ln(1+e^{-2})\right]$$

$$= 0.18312 \leftarrow p$$

$\therefore$ Binomial $E(X) = \mu = np = 30 \cdot 0.18312$

$V(X) = np(1-p) = 4.4876.$

4.3 Cumulative Distribution Function.

(C.D.F)

C.D.F: X is Continuous R.V.

C.D.F: X is Continuous R.V.

C.D.F of X is defined as

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(u)\,du.$$

$$\left(\sum_{u<x} f(u)\,du\right)$$

$$\int_{-\infty}^{0} f(x)\,dx = 0.$$

Example: p.d.f

$$f(x) = 0.01\,e^{-0.01x}, \quad x \ge 0.$$

Then Find the C.D.F of X.

$$F(x) = P(X \le x) = \int_0^x 0.01\,e^{-0.01x}\,dx$$

$$= -e^{-0.01x}\Big|_0^x$$

$$= 1 - e^{-0.01x}$$

$x = 200$

$$\therefore P(X < 200) = 1 - e^{-0.01 \cdot 200} = 0.8647$$

P.D.F and C.D.F

$$f(x) = \frac{dF(x)}{dx}$$

Example:

$$f(x) = \frac{2}{\ln 2}\,\frac{e^{-2(x-4)}}{1+e^{-2(x-4)}}, \quad x \ge 4.$$

C.D.F. ?

$$F(x) = \int_{-\infty}^{4} f(x)\,dx + \int_4^x f(x)\,dx$$

$$= 0 + \frac{2}{\ln 2}\left[-\frac{1}{2}\ln\left[1+e^{-2(x-4)}\right]\right]\Big|_4^x$$

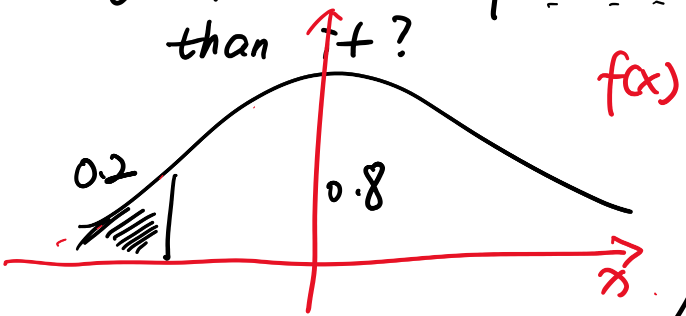
$$= 1 - \frac{1}{\ln 2}\cdot \ln\left[1+e^{-2(x-4)}\right]$$

$$\Rightarrow \quad = 1 - \frac{1}{\ln 2}\cdot \ln\left[1+e^{-2(x-4)}\right], \quad x \geq 4$$

$$F(x=\infty) = 1$$
$$= P(X \leq \infty) = 1$$

Fill the blank:

what value of x has 80% of total greater than it?



$$F(X \leq x) = 0.2$$

$$1 - \frac{1}{\ln 2}\left[\ln(1+e^{-2(x-4)})\right] = 0.2$$

$$\Longleftrightarrow \quad x = 4.1498$$

4.4 Mean and Variance for Continuous R.V.

∴ Similar to Discrete ones. $\Sigma \to \int$

$$E(X) = \mu = \int_{-\infty}^{\infty} x f(x)\, dx.$$

$$V(X) = \sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx = E(X^2) - [E(X)]^2$$

S.D: $\sigma = \sqrt{\sigma^2}$

Example:

$$F(x) = \begin{cases} 0 & x < 3 \\ \frac{1}{27}(x^2 - 9) & 3 \leq x < 6 \\ 1 & x \geq 6 \end{cases} \qquad (x^2)' = 2x$$

Find Mean and Variance for X.

Find Mean and Variance for X.

$$f(x) = \frac{dF(x)}{dx} \qquad f(x) = \begin{cases} 0 & x < 3 \\ \frac{2}{27}x & 3 \le x < 6 \\ 0 & x \ge 6 \end{cases}$$

$$\therefore E(x) = \mu = \int_3^6 x\, f(x)\, dx$$
$$= \int_3^6 \frac{2}{27}x^2\, dx.$$
$$= \frac{2}{27} \cdot \frac{1}{3}x^3 \Big|_3^6$$
$$= \frac{14}{3}$$

$$V(X) = \int (x-\mu)^2 f(x)\, dx$$
also ok.

$$E(x^2) = \int_3^6 x^2 f(x)\, dx$$
$$= \int_3^6 \frac{2}{27}x^3\, dx$$
$$= \frac{2}{27} \cdot \frac{1}{4}x^4 \Big|_3^6$$
$$= 22.5$$

$$\therefore V(x) = E(x^2) - [E(x)]^2 = 0.7222$$

Expected Value of a Function of Continuous R.V.

$$\therefore E[h(x)] = \int h(x) f(x)\, dx.$$

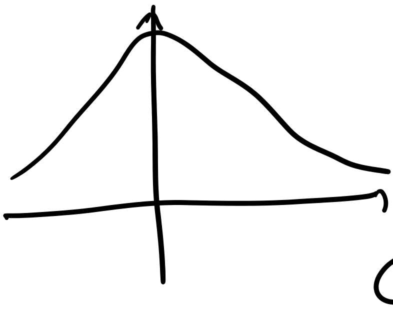
$$h(x) = x^2 \qquad E(x^2) = \int x^2 f(x)\, dx.$$

4.6 Normal Distribution.

1. Law of Large Number.
2. Central Limit Theorem.

2. Central Limit Theorem.

whenever a random experiment is replicated, the R.V that equals the averge of experiments tends to have a normal distribution, as the replicates become large.



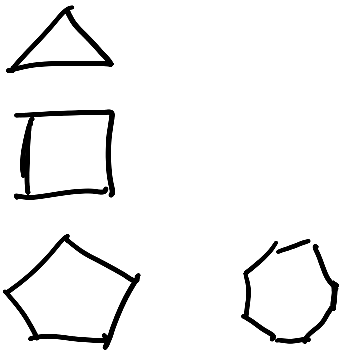
Bell curve.

De Moive. 1733

Gaussian Distribution

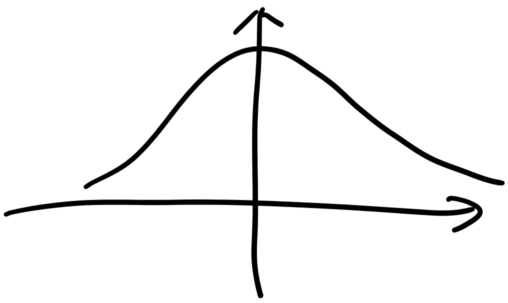
Gauss

17 side polgon



Normal Distribution:

A random varible X



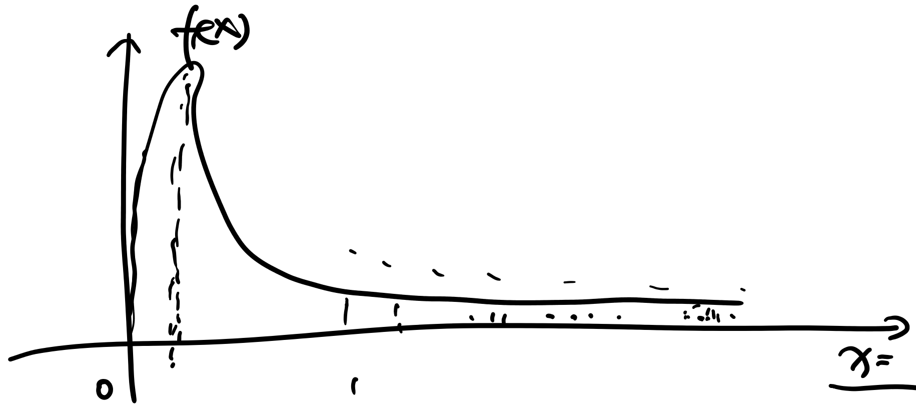
with p.d. f:

$$f(x) = \frac{1}{\sqrt{2\pi}\cdot\sigma}\cdot e^{\frac{-(x-\mu)^2}{2\sigma^2}}, \quad x \in (-\infty, +\infty).$$

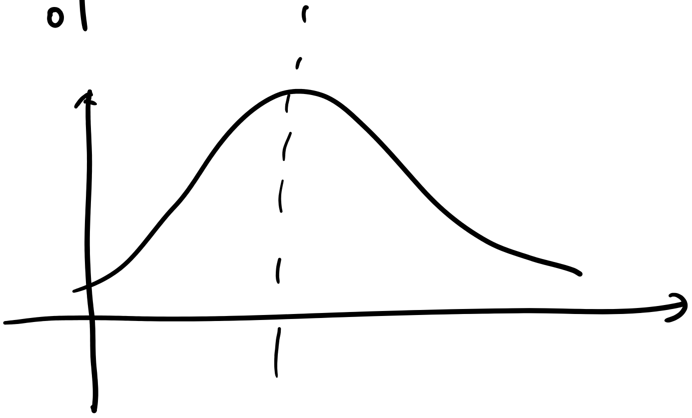
then, it is a normal R.V with parameter μ and σ, where $\mu \in (-\infty, +\infty)$, $\sigma > 0$.

$\therefore E(X) = \mu$. $V(X) = \sigma^2$.

$\therefore \; E(X) = \mu \; , \quad V(X) = \sigma^2 .$

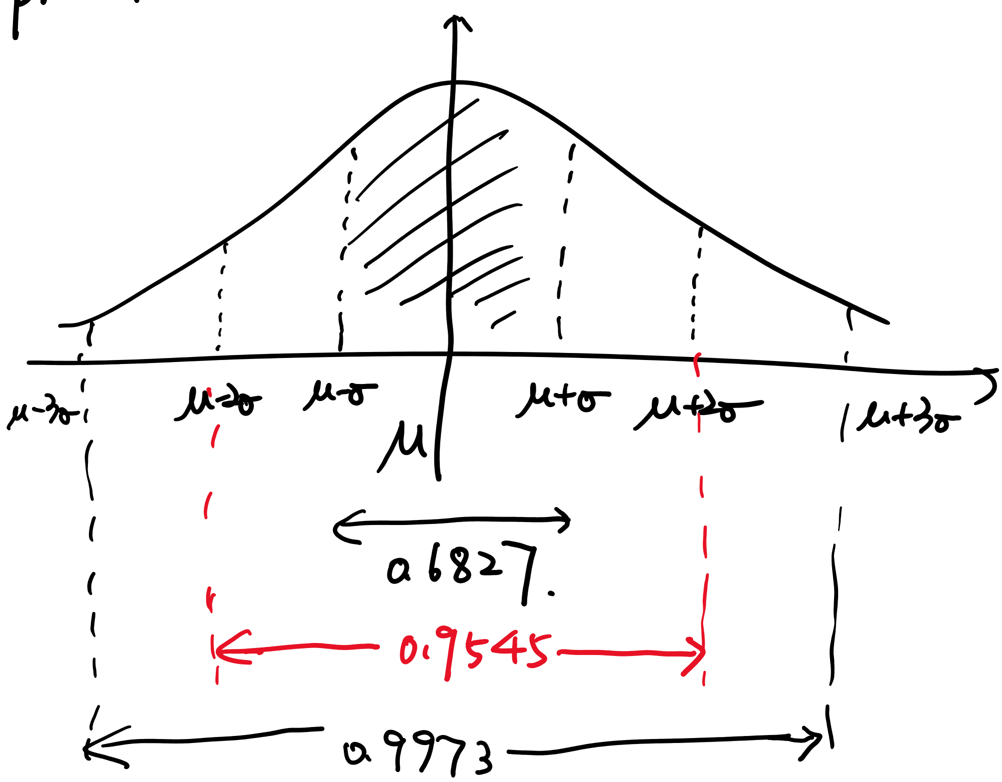


x = Income for Candian.



$\bar{x}$ = Average Income

Emprical Rule:



$\mu = 170$ cm

$\sigma = 10 \text{ cm}^2 .$

$\therefore$ 99.73% of human being will have height between 140, 200 cm

Standard normal distribution:

Standard normal distribution:

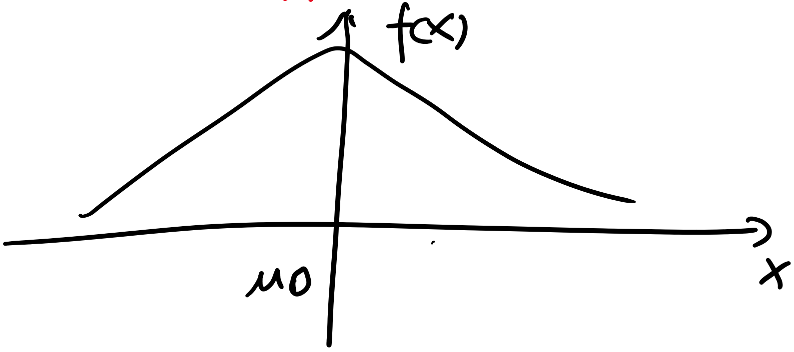
A normal R.V with $\mu=0$, $\sigma^2=1$ is called a standard normal R.V, denoted by Z.

standard normal

Then C.D.F. of Z:

is $\Phi(z) = P(Z \le z)$

standard normal



$\mu=0$
$\sigma^2=1$

(1) $P(Z > 1.26)$

$= 1 - P(Z \le 1.26)$

$= 1 - 0.8962$

$= 0.1038$

(2) $P(Z > -1.37)$

① $= 1 - P_r(Z \le -1.37)$

② $= P(Z < 1.37) = 0.9147$

$= 1 - 0.0853$

$= 1 - 0.0853$

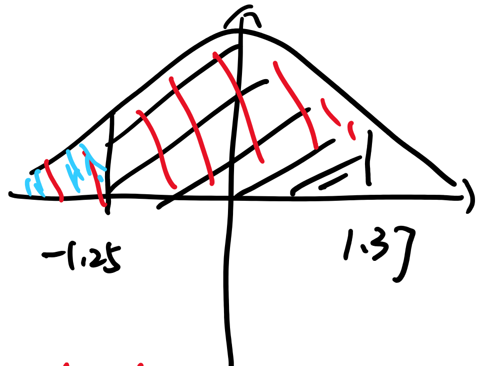
$= 0.9147$

(3) $P(-1.25 < Z < 1.37)$

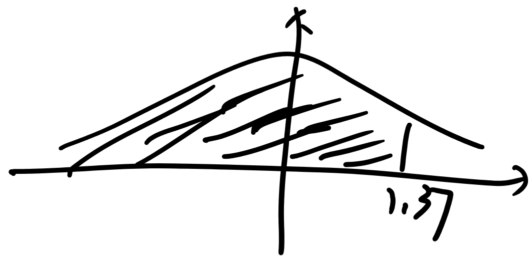
$= \underline{P(Z < 1.37)} - \underline{P(Z < -1.25)}$

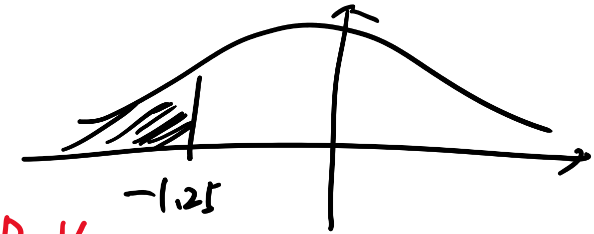
$= 0.9147 - 0.1056$

$= 0.8991$



Red - Blue





Standizing a Normal R.V.

If X is a normal R.V. with $E(X) = \mu$ and $V(X) = \sigma^2$, then

$$\boxed{Z = \frac{X - \mu}{\sigma}}$$

is a standard normal R.V

$\therefore Z : \begin{cases} E(Z) = 0 \\ V(Z) = 1. \end{cases}$

Probability:

$$P(X \le x) = P\left(\frac{X-\mu}{\sigma} < \frac{x-\mu}{\sigma}\right)$$

$$= Pr(Z < z) \leftarrow \text{use } Z \text{ table.}$$

$= Pr(Z < z)$ ← use Z table.

Example:

Suppose the diameter of a shaft in an optical storage is normally distributed with mean 0.2508 (μ) and $\sigma = 0.0005$ inches.

Then, what's the probability that a random shaft is within 0.2485 and 0.2515?

$$P(0.2485 < \boxed{X} < 0.2515)$$

$$= P\left(\frac{0.2485 - 0.2508}{0.0005} < \underbrace{\frac{X - \mu}{\sigma}}_{Z} < \frac{0.2515 - 0.2508}{0.0005}\right)$$

$$= P(-4.6 < Z < 1.4)$$

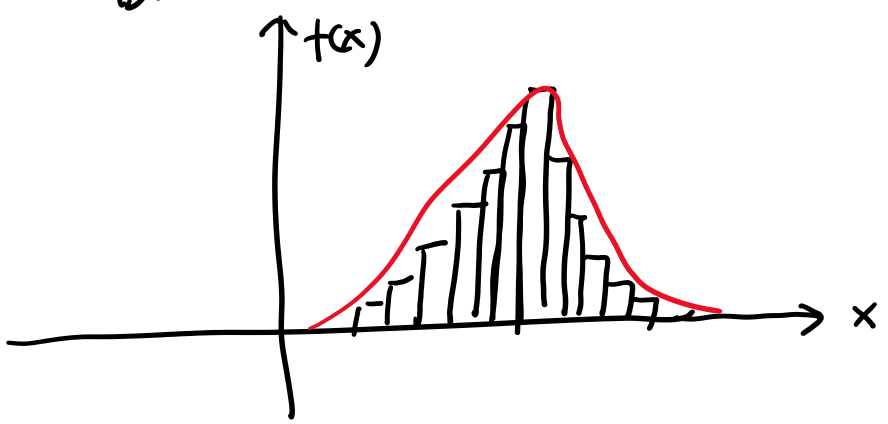
$$= P(Z \le 1.4) - P(Z \le -4.6)$$

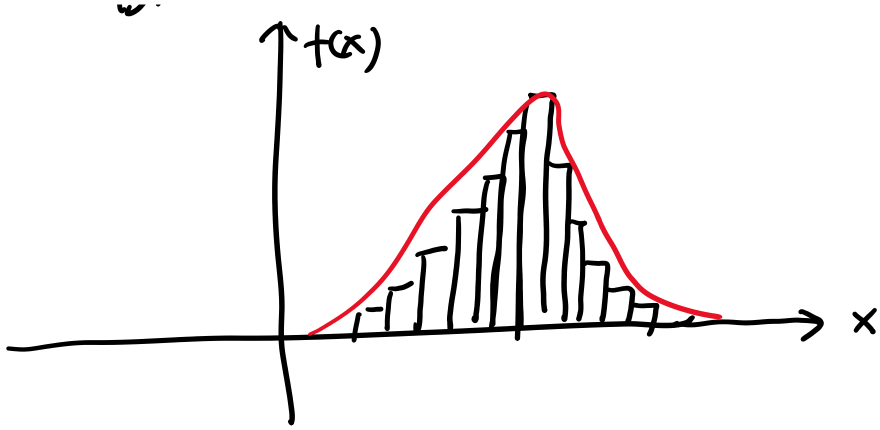
$$= 0.9192 - 0$$

$$= 0.9192$$

Test 1.

4.7 Normal Approximation to Binomial and Poisson distribution:





Example: Consider in a digital communication channel. # of bits received in error can be modoled a binomial R.V.

Assume the probability that a bit is received in error is 10^{-5}.

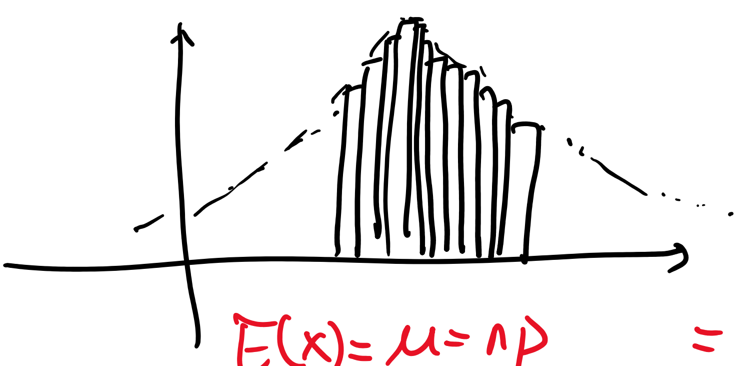
However, 16,000,000 bits are transmited. what's probability that 150 or fewer error occur?

Try Binomial.

$n = 16,000,000$

$p = 10^{-5}$

$$P(X \le 150) = \sum_{x=0}^{150} \binom{16\,000\,000}{x} (10^{-5})^{x} (1-10^{-5})^{16000000-x}$$



as n is Large.

$E(X) = \mu = np \quad = 160$

$V(X) = \sigma^2 = np(1-p) = 160(1-10^{-5})$

$P(X \le 150) = P_r\left(\frac{X-\mu+0.5}{} < \frac{150-160+0.5}{} \right)$

$$P(X \le 150) = Pr\left(\frac{X-\mu+0.5}{\sigma} < \frac{150-160+0.5}{\sqrt{160(1-10^{-5})}}\right)$$

$$= Pr(Z < -0.75)$$

$$= 0.2266.$$

If X is a binomial R.V. with n and p.

then.

$$Z = \frac{X-np}{\sqrt{np(1-p)}}$$

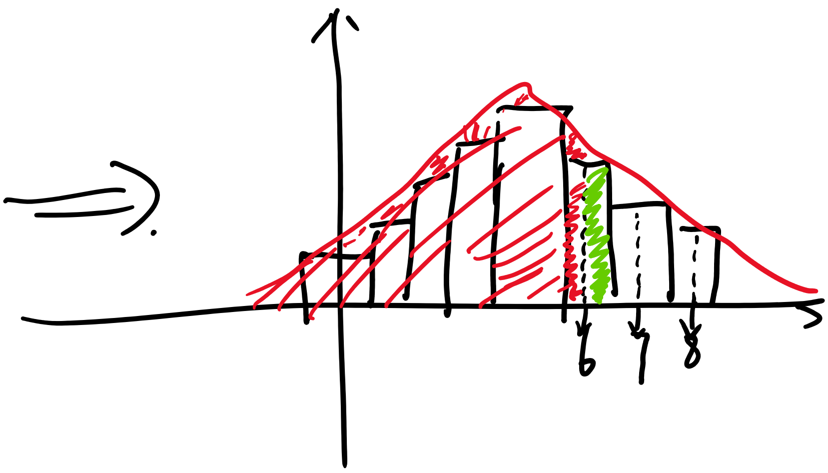
is approximately a standard normal R.V.

To approximate its probability, a continuity correction is applied as following:

(1) $P(X \le x) = P(X \le x+0.5) \approx P\left(Z \le \frac{x+0.5-np}{\sqrt{np(1-p)}}\right)$

(2) $Pr(X \ge x) = P(X \ge x-0.5) \approx P\left(\frac{x-0.5-np}{\sqrt{np(1-p)}} \le Z\right)$

$\Rightarrow$



$Pr(X \le 6)$

$= Pr(X \le 6.5)$

Poisson Approximation:

Poisson Approximation:

If X is Poisson R.V.

$E(X) = \lambda$, $V(X) = \lambda$

Then

$Z = \frac{X-\lambda}{\sqrt{\lambda}}$ is same with pevious one.

Both approximation, $\left\{ \begin{array}{l} E(x) = np > 5 \\ n(1-p) > 5 \end{array} \right.$

$\{ \lambda > 5$

The approximation is good.

Example: Poisson $\lambda = 1000$.

$$P(X \le 950) = \sum_{x=0}^{950} \frac{e^{-1000} 1000^x}{x!}$$

$\lambda = 1000 > 5$

Approximate use Normal:

$$P(X \le 950) = P(X < 950.5)$$

$$= P\left(Z < \frac{950.5 - 1000}{\sqrt{1000}}\right)$$

$$= P(Z < -1.57)$$

$$= 0.0582.$$

$$P(X=5) = P(4.5 \le X \le 5.5)$$

4.8 Expontial Distribution:

Poisson : λ : # mean of success for a cetain time.

Expontial: the next success will occur in a time period $\frac{1}{\lambda}$ on average.

P.D.F for expontial:

$$f(x) = \lambda e^{-\lambda x}, \qquad 0 \le x < \infty$$

C.D.F for expontial:

$$F(x) = 1 - e^{-\lambda x}, \qquad 0 \le x < \infty.$$

$$\left(\begin{aligned} F(x) &= \int_0^x \lambda e^{-\lambda x}\, dx \\ &= e^{-\lambda x} \Big|_0^x \\ &= 1 - e^{-\lambda x} \end{aligned} \right)$$

Mean and Variance:

$$\boxed{E(X)} = \int_0^\infty x \cdot \lambda e^{-\lambda x}\, dx \qquad \left(\int u\,dv = uv - \int v\,du\right)$$

$$= \lambda \left[\frac{-xe^{-\lambda x}}{\lambda} \Big|_0^\infty + \frac{1}{\lambda} \int_0^\infty e^{-\lambda x}\, dx \right]$$

$$= \lambda \left[0 + \frac{1}{\lambda} \cdot \frac{-e^{-\lambda x}}{\lambda} \Big|_0^\infty \right]$$

$$= \lambda \cdot \frac{1}{\lambda^2} = \boxed{\frac{1}{\lambda}}$$

$$E(X^2) = \cdots\cdot$$

$$= \frac{2}{\lambda^2}$$

$$\boxed{V(X)} = E(X^2) - [E(X)]^2 = \boxed{\frac{1}{\lambda^2}}$$

Memoryless property:

Example: Now cpu of a personal computer lifetime exponential ($\frac{1}{\lambda} = 6$ years).

$$\therefore \lambda = \frac{1}{6}$$

(a) You have owned cpu 4 years, what's the probability it will fail within next year?

$$P(\underline{X<5} \mid \underbrace{X>4}_{\text{use 4 years}}) = \frac{P(4<X<5)}{P(X>4)}$$

$$f(x) = \frac{1}{6} e^{-\frac{1}{6}x}$$

$$F(x) = P(X \leq x) = 1 - e^{-\frac{1}{6}x}$$

CDF CDF

$$= \frac{P(X<5) - P(X<4)}{1 - P(X<4)}$$ ← CDF

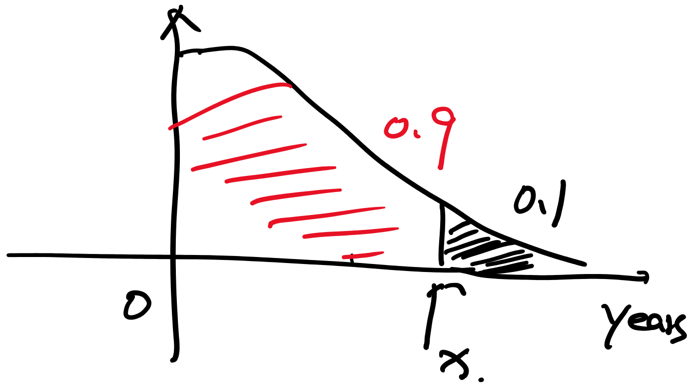
$$= \frac{1 - e^{-\frac{1}{6}\cdot 5} - (1 - e^{-\frac{1}{6}\cdot 4})}{1-(1-e^{-\frac{1}{6}\cdot 4})}$$

$$= 1 - e^{-\frac{1}{6}}$$

whats probability a cpu will fail in a year?

$$P(X<1) = 1 - e^{-\frac{1}{6}\cdot 1} = 1 - e^{-\frac{1}{6}}$$

(b). 10% cpus last longer than how many years?



$$P(X < x) = 0.9.$$

$$\therefore 1 - e^{-\frac{1}{6}x} = 0.9.$$

$$x = 13.82. \text{ years.}$$

(c) If you buy 10 cpus. find the probability that they all last longer than 7 years.

Single one

$$P(X > 7) = 1 - P(X \le 7)$$

$$= 1 - (1 - e^{-\frac{1}{6}\cdot 7})$$

$$= e^{-\frac{7}{6}}$$

Binomial $n = 10$

Y $p = e^{-\frac{7}{6}}$

$P(Y = 10)$

$$\left(e^{-\frac{7}{6}}\right)^{10} = 8.574931 \times 10^{-6}$$

(d). If a company buys 500 CPUs.

$\boxed{\text{approximately}}$ probability of Between 100 and 150 of

$\boxed{\text{approximately}}$, probability of (Between 100 and 150 of them surive over 7 years)

$n = 500$.

$p = e^{-\frac{7}{6}}$

Binomial $\Rightarrow$ $np = 500\, e^{-\frac{7}{6}} > 5$

$n(1-p) = 500(1 - e^{-\frac{7}{6}}) > 5$

$P(100 \le X \le 150)$ $\leftarrow$ Normal Approximation

$= P\left(\frac{99.5 - \mu}{\sigma} \le Z \le \frac{150.5 - \mu}{\sigma}\right)$

where $\mu = np = 500 \cdot e^{-\frac{7}{6}}$

$\sigma = \sqrt{np(1-p)} = \sqrt{500 \cdot e^{-\frac{7}{6}} \cdot (1 - e^{-\frac{7}{6}})}$

$= P(-5.43 \le Z \le -0.5)$

$= P(Z \le -0.5) - P(Z \le -5.43)$

$= 0.3085 - 0$

$= 0.3085$